NTNU | Department of Mathematical Sciences

EXPLORING GENERALIZATIONS OF τ-TILTING

Related structures and higher settings

Endre Sørmo Rundsveen

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Contents

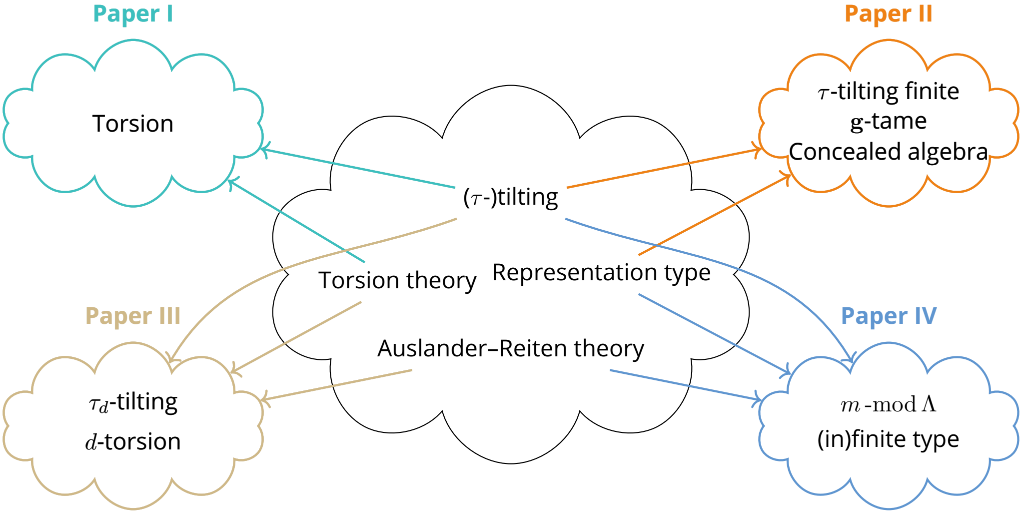
Paper I
Universal extensions and Ext-orthognal complements of torsion classes

Paper II
τ-tilting finiteness and g-tameness: Incidence algebras of posets and concealed algebras

Paper III
τ_d-tilting theory for linear Nakayama algebras

Paper IV
A note on the extended module categories of homogeneous Nakayama algebras

Overview



Paper I
Torsion
Paper II
τ-tilting finite
g-tame
Concealed algebra
(τ-)tilting
Torsion theory
Representation type
Auslander–Reiten theory
Paper III
τ_d-tilting
d-torsion
Paper IV
m-mod Λ
(in)finite type

Paper I

UNIVERSAL EXTENSIONS AND EXT-ORTHOGNAL COMPLEMENTS OF TORSION CLASSES

Endre S. Rundsveen

Torsion pairs

Definition **[Dickson '66]**

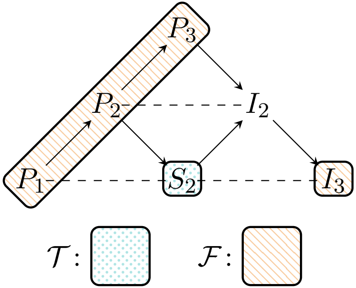
A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of an abelian category $\mathcal{A}$ is a *torsion pair* if

1. $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$, and
2. for each $X \in \mathcal{A}$ there exists a short exact sequence

$$0 \longrightarrow \mathfrak{t}X \longrightarrow X \longrightarrow \mathfrak{f}X \longrightarrow 0$$

   such that $\mathfrak{t}X \in \mathcal{T}$ and $\mathfrak{f}X \in \mathcal{F}$.

Example



Previously known

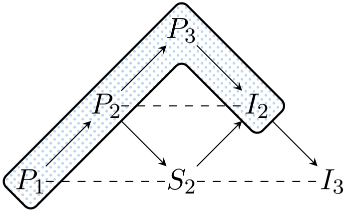
Teorem **[Bauer–Botnan–Oppermann–Steen'20, Buan–Zhou'24]**

Let $(\mathcal{T}, \mathcal{F})$ be a functorially finite torsion pair. Then we have the following equivalence

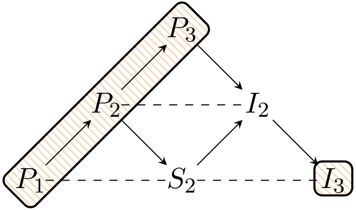
$$\frac{{}^{\perp_E}\mathcal{T}}{{}^{\perp_E}\mathcal{T} \cap \mathcal{T}} \simeq \mathcal{F}$$

[Bauer–Botnan–Oppermann–Steen'20] - A result on tilting
[Buan–Zhou'24] - A result on τ-tilting

Example cont'd



$^{\perp_E}\mathcal{T} \setminus \mathcal{T}$



$\mathcal{F}$

Motivating question

Can we remove the assumption of functorially finiteness?

Answer

Teorem [R.'24]

Let $\mathcal{A}$ be an abelian Krull–Schmidt category and $(\mathcal{T}, \mathcal{F})$ a torsion pair of $\mathcal{A}$. There is an equivalence

$$\frac{{}^{\perp_E}\mathcal{T}}{\mathcal{I}} \simeq \mathcal{E}$$

where $\mathcal{E}$ is the subcategory of objects in $\mathcal{F}$ with a universal extension to $\mathcal{T}$, and $\mathcal{I}$ is the ideal in ${}^{\perp_E}\mathcal{T}$ of morphisms factoring through $\mathcal{T}$.

Note, this can also be seen as a consequence of a result by [Demonet–Iyama'16].

Paper II

τ-TILTING FINITENESS AND g-TAMENESS: INCIDENCE ALGEBRAS OF POSETS AND CONCEALED ALGEBRAS

Erlend D. Børve   Jacob Fjeld Grevstad
Endre S. Rundsveen

Definitions

Definition [Adachi–Iyama–Reiten'14]

A pair (M, P) of finitely generated Λ-modules is a *τ-rigid pair* if

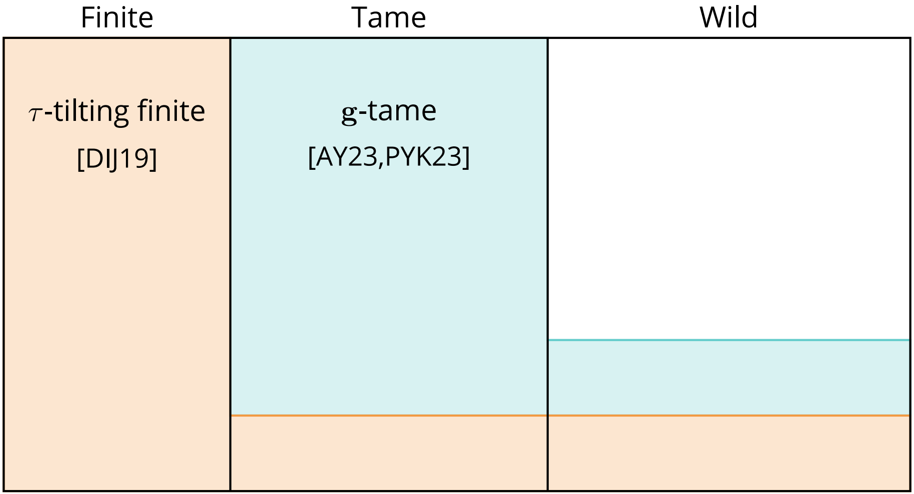
1. M is *τ-rigid*, i.e. $\mathrm{Hom}_\Lambda(M, \tau M) = 0$, and
2. P is projective such that $\mathrm{Hom}_\Lambda(P, M) = 0$.

Moreover, a τ-rigid pair (M, P) is *τ-tilting* if $|M| + |P| = |\Lambda|$.

Definition

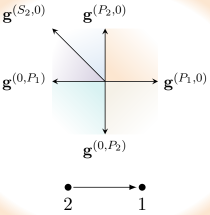
Let $(P, \leq)$ be a poset with Hasse quiver Q. The *incidence algebra* of $(P, \leq)$ is given by $\mathbf{k}Q/I$, where I is generated by commutivity relations.

How many?



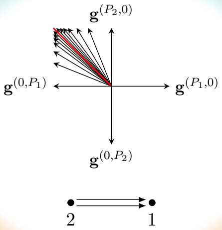
How many?

Finite	Tame	Wild
τ-tilting finite [DIJ19]	g-tame [AY23,PYK23]	

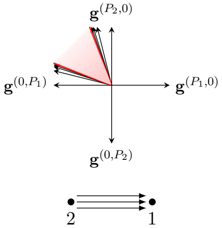


How many?

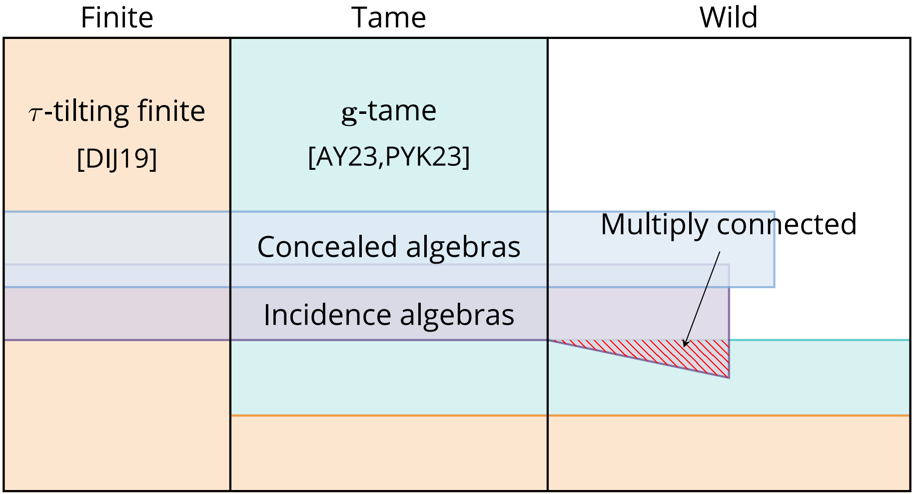
Finite	Tame	Wild
τ-tilting finite [DIJ19]	g-tame [AY23,PYK23]	



How many?

Finite	Tame	Wild
τ-tilting finite [DIJ19]	g-tame [AY23,PYK23]	

Results

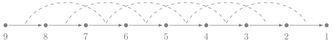

Finite
Tame
Wild
τ-tilting finite
[DIJ19]
g-tame
[AY23,PYK23]
Multiply connected
Concealed algebras
Incidence algebras

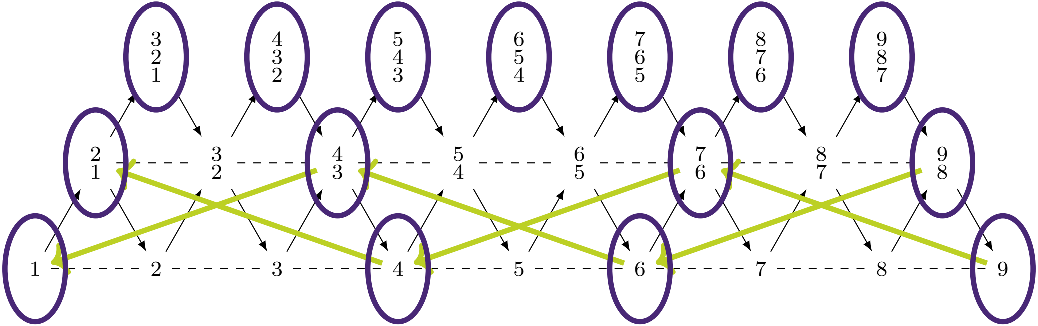
Paper III

τ_d-TILTING THEORY FOR LINEAR NAKAYAMA ALGEBRAS

Endre S. Rundsveen Laertis Vaso

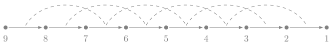
d-cluster tilting subcategory

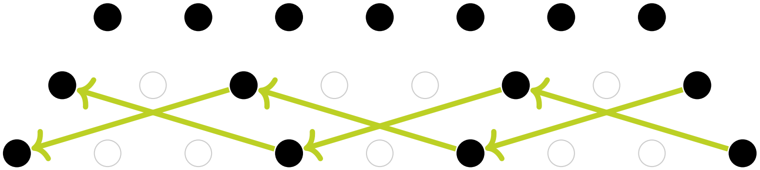




2-cluster tilting subcategory of $\Lambda(9, 4)$.

d-cluster tilting subcategory





d-cluster tilting subcategory – Definitions

Definition **[Iyama'07]**

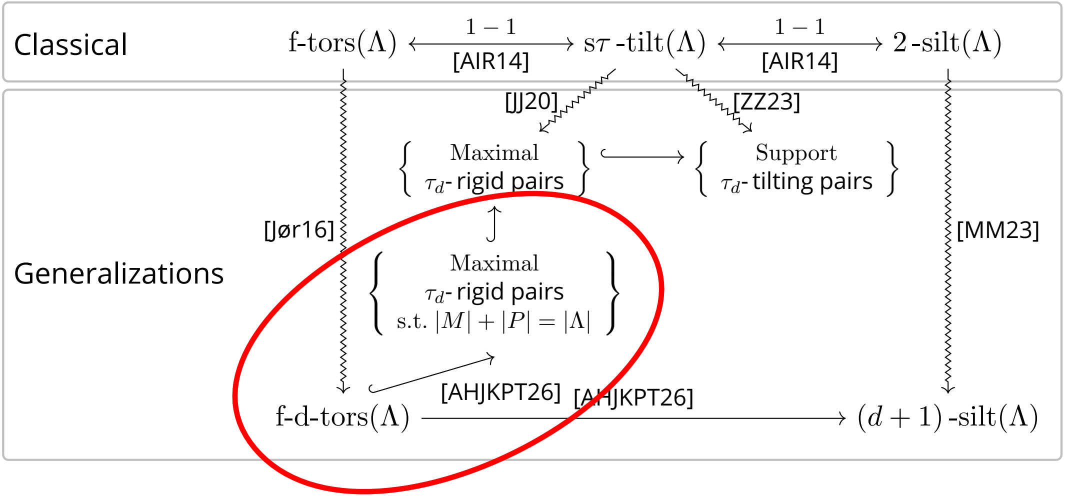
Let Λ be a finite dimensional $\mathbf{k}$-algebra. A functorially finite subcategory $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ is a *d-cluster tilting subcategory* if

$$\begin{aligned}\mathcal{C} &= \{\, M \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^i_\Lambda(M, \mathcal{C}) = 0 \text{ for } 1 \leq i \leq d-1 \,\} \\ &= \{\, M \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^i_\Lambda(\mathcal{C}, M) = 0 \text{ for } 1 \leq i \leq d-1 \,\}\end{aligned}$$

The d-Auslander Reiten translates are given by

$$\tau_d(M) := \tau\Omega^{d-1}(M) \qquad \tau_d^-(M) := \tau^-\Omega^{-(d-1)}(M)$$

Generalizations



d-cluster tilting subcategory – Algebra of interest

An arbitrary algebra Λ is not guaranteed to admit a d-cluster tilting subcategory.

Focus on $\Lambda(n,l) = \mathbf{k}\mathbb{A}_n/\langle \text{paths of length } l \rangle$ for $l \geq 2$.

$$\mathbb{A}_n : \quad \underset{n}{\bullet} \longrightarrow \underset{n-1}{\bullet} \longrightarrow \cdots \longrightarrow \underset{3}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{1}{\bullet}$$

Classified by [Vaso´18] when these admit d-cluster tilting subcategories.

τ_d-rigidness – Definitions

Definition [Adachi–Iyama–Reiten'14]

A pair (M, P) of finitely generated Λ-modules is a τ-*rigid pair* if

1. M is τ-*rigid*, i.e. $\mathrm{Hom}_\Lambda(M, \tau M) = 0$, and
2. P is projective such that $\mathrm{Hom}_\Lambda(P, M) = 0$.

Moreover, a τ-rigid pair (M, P) is τ-*tilting* if $|M| + |P| = |\Lambda|$.

τ_d-rigidness – Definitions

Definition [Jacobsen–Jørgensen'20]

Let $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ be a d-cluster tilting subcategory.

A pair (M, P) of finitely generated Λ-modules with $M \in \mathcal{C}$ is a τ_d-*rigid pair* if

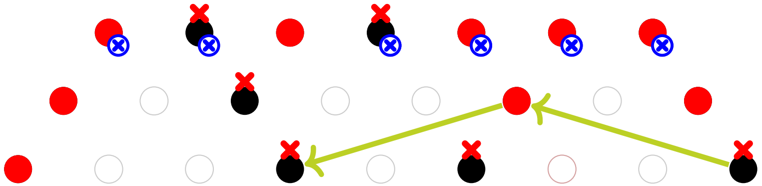
1. M is τ_d-*rigid*, i.e. $\operatorname{Hom}_\Lambda(M, \tau_d M) = 0$,
2. P is projective such that $\operatorname{Hom}_\Lambda(P, M) = 0$.

Moreover, a τ_d-rigid pair (M, P) is a *summand maximal* τ_d-*rigid pair* if it is τ_d-rigid and for any τ_d-rigid pair (N, Q) we have

$$|N \oplus Q| \leq |M \oplus P|$$

τ_d-rigidness – Example Cont'd

Finding τ_2-rigid pair (M, P)



τ_d-rigidness – Strategy

Aim for *local* maximality!

τ_d-rigidness – Local maximality

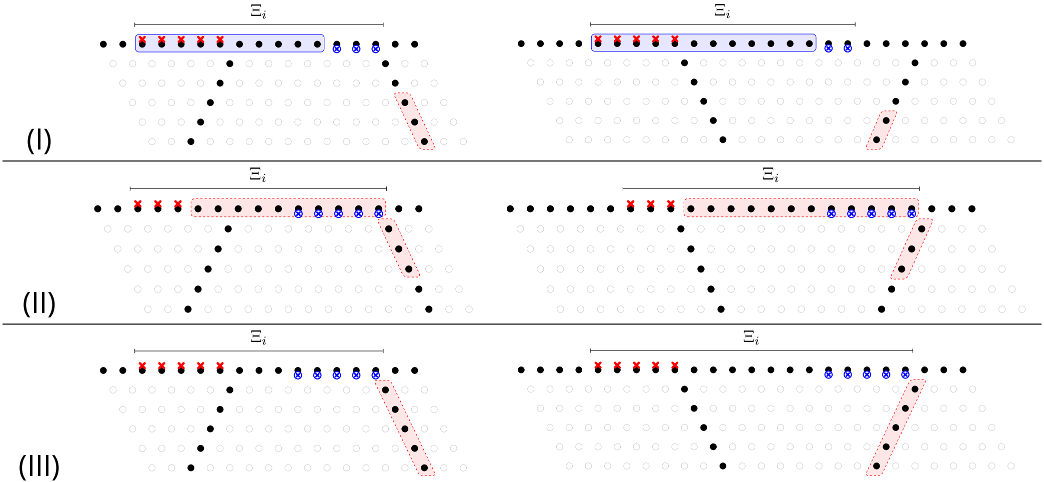


Table: Admissible configurations

τ_d-rigidness – Local maximality

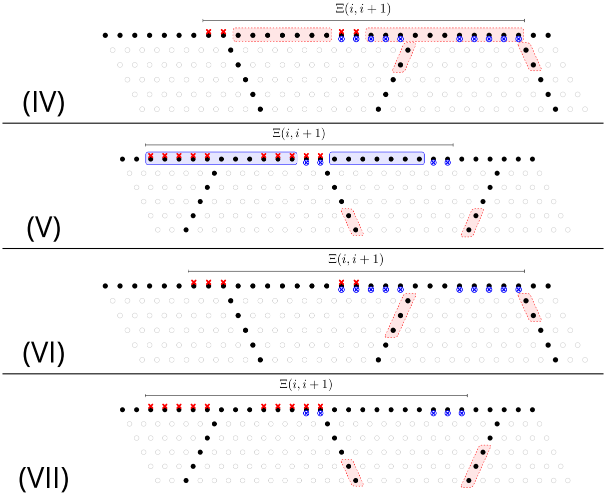


Table: Admissible configurations

τ_d-rigidness – Local maximality for $d = 2$

(VIII)

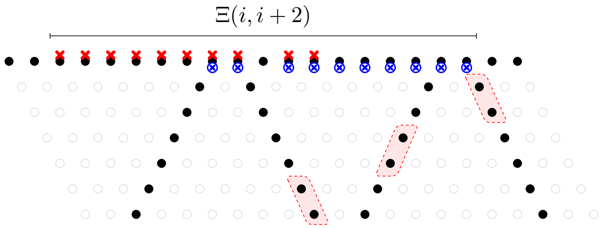


Table: Admissible configurations

τ_d-rigidness – Connect to one whole

T_i \ T_{i+1}		I	II	III	IV	V	VI R	VI SR	VII	VIII
I		S		S		S		S	S	S
II		A	R	A	R	A	R	A	A	A
III		A	R	A	R	A	R	A	A	A
IV		A	R	A	R	A	R	A	A	A
V		S		S		S		S	S	S
VI		A	R	A	R	A	R	A	A	A
VII	S	S		S		S		S	S	S
VII	SR	A	R	A	R	A	R	A	A	A
VIII		A	R	A	R	A	R	A	A	A

Table: Which components can be connected and how.

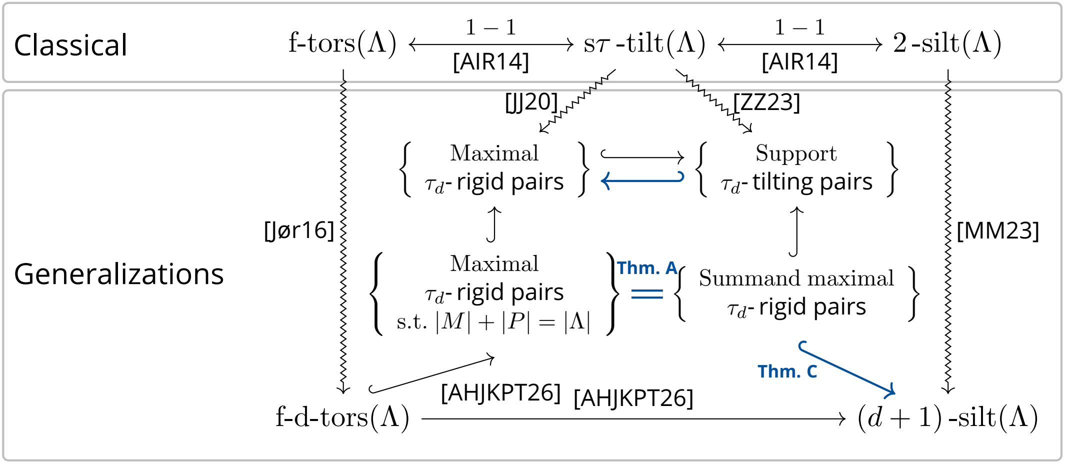
τ_d-rigidness – Result

Teorem **[Theorem A, R.–Vaso '25]**

Assume $\Lambda = \Lambda(n, l)$ admits a d-cluster tilting subcategory $\mathcal{C}$ and let $(M, P) \in \mathcal{C} \times \operatorname{proj} \Lambda$ be a basic pair. TFAE

1. (M, P) is a summand maximal τ_d-rigid pair,
2. (M, P) is *well-configured*,
3. (M, P) is τ_d-rigid and $|M| + |P| = |\Lambda|$.

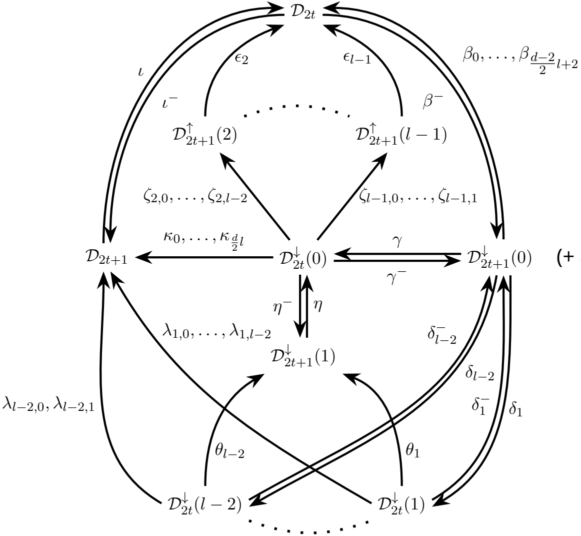
Generalizations



For $\Lambda(n,l)$.

d-torsion classes – Result

The graph G:



(+ additional arrows for $d = 2$)

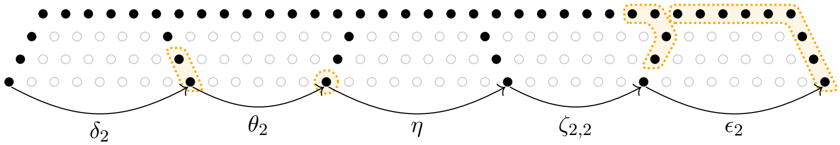
d-torsion classes – Result

Teorem **[Theorem B, R.–Vaso'25]**

Let $\mathcal{C} \subseteq \operatorname{mod} \Lambda(n, l)$ be a d-cluster tilting subcategory. There is a bijection

$$\left\{ \text{d-torsion classes in } \mathcal{C} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{Paths in } G \text{ of length} \\ p-1 \text{ starting in odd vertex.} \end{array} \right\}$$

d-torsion class – Example



Paper IV

A NOTE ON THE EXTENDED MODULE CATEGORIES OF HOMOGENEOUS NAKAYAMA ALGEBRAS

Endre S. Rundsveen

Another direction

Classical

$$\text{f-tors}(\Lambda) \xleftrightarrow[\text{[AIR14]}]{1-1} \text{s}\tau\text{-tilt}(\Lambda) \xleftrightarrow[\text{[AIR14]}]{1-1} 2\text{-silt}(\Lambda)$$

Generalizations

$$2\text{-silt}(\Lambda) \rightsquigarrow (\mathrm{m}+1)\text{-silt}\,\Lambda$$

Extended module categories

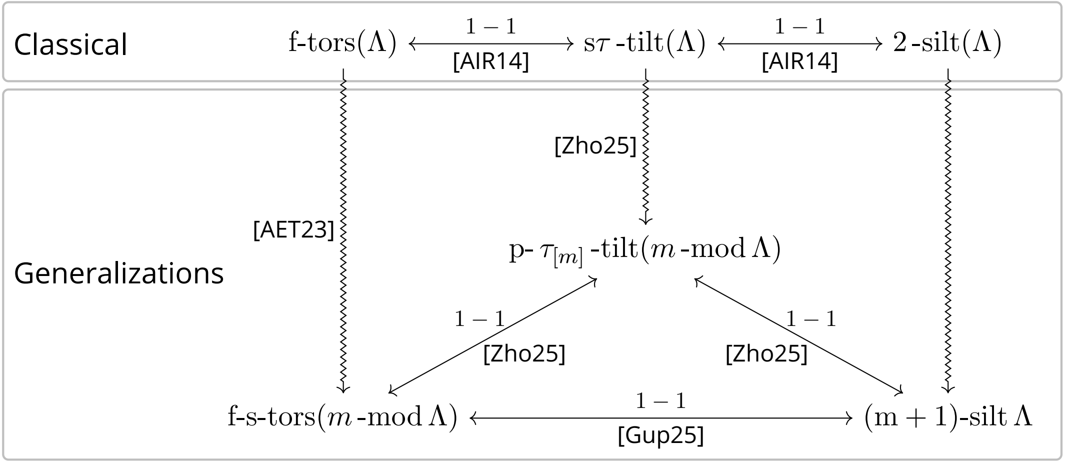
$$\frac{\mathbf{K}^{[-1,0]}(\operatorname{proj}\Lambda)}{\Lambda[1]} \cong \operatorname{mod}\Lambda$$

$$\begin{aligned}\frac{\mathbf{K}^{[-m,0]}(\operatorname{proj}\Lambda)}{\Lambda[m]} &\cong \{\, X^\bullet \in \mathbf{D}^b(\operatorname{mod}\Lambda) \mid H^i(X^\bullet) = 0 \text{ for } i \notin [-(m-1), 0] \,\} \\ &= m\text{-}\operatorname{mod}\Lambda\end{aligned}$$

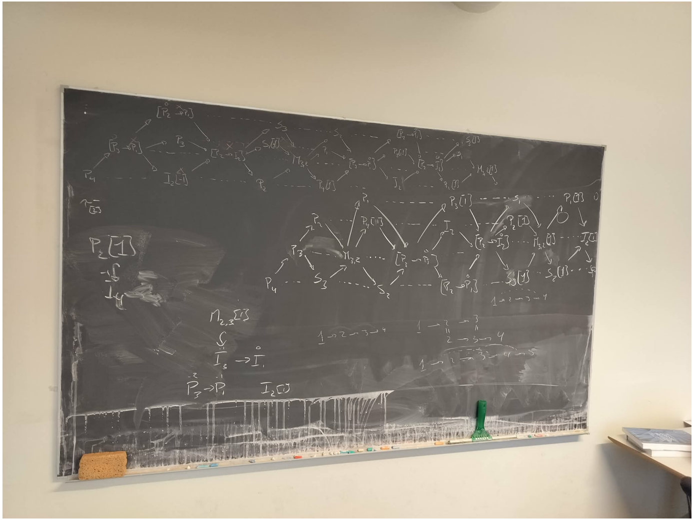
Objects are complexes instead of modules.

$$\cdots \longrightarrow X^{-(m-1)} \xrightarrow{d_X^{-(m-1)}} X^{-(m-2)} \longrightarrow \cdots \longrightarrow X^{-1} \xrightarrow{d_X^{-1}} X^0 \longrightarrow \cdots$$

Another direction



m-mod Λ – Preliminary exploration



$m\text{-mod}\,\Lambda$ – Initial question

How do the extended module categories behave and look like for Nakayama algebras?

m-mod Λ – Knitting

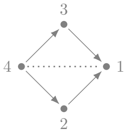
$$\begin{array}{ccccccccc}
\cdots \longrightarrow & X^{-(m-1)} & \xrightarrow{d_X^{-(m-1)}} & X^{-(m-2)} & \longrightarrow \cdots \longrightarrow & X^{-1} & \xrightarrow{d_X^{-1}} & X^{0} & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& \underline{\dim}(H^{-(m-1)}(X^\bullet)) & & \underline{\dim}(H^{-(m-2)}(X^\bullet)) & & \underline{\dim}(H^{-1}(X^\bullet)) & & \underline{\dim}(H^{0}(X^\bullet)) &
\end{array}$$

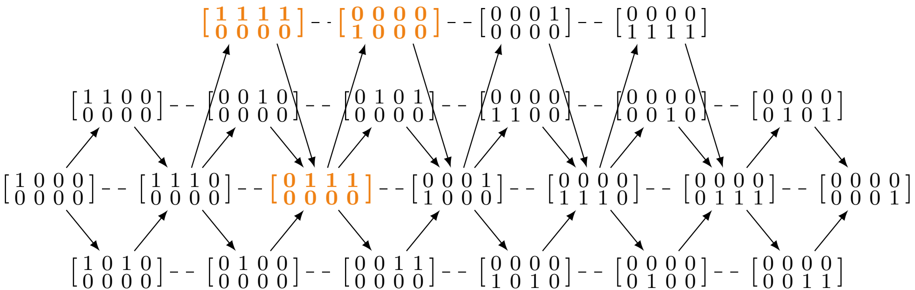
Definition

The *cohomological dimension vector* of $X^\bullet \in m\text{-mod}\,\Lambda$ is given by the sequence

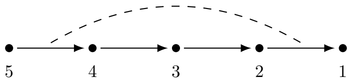
$$\underline{\mathrm{Dim}}(X^\bullet) := (\underline{\dim} H^i(X^\bullet))_{-(m-1)\leq i\leq 0}$$

m-mod Λ – Knitting

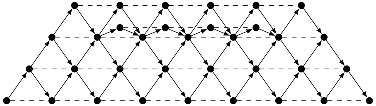




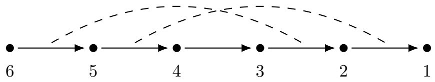
$2\text{-mod}\,\Lambda(5,4)$ – Python-assisted exploration



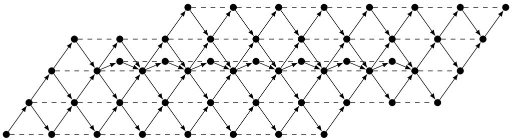
The AR-quiver of $2\text{-mod}\,\Lambda(5,4)$ is given by



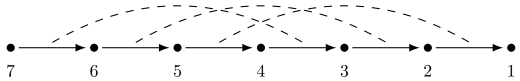
$2\text{-mod}\,\Lambda(6,4)$ – Python-assisted exploration



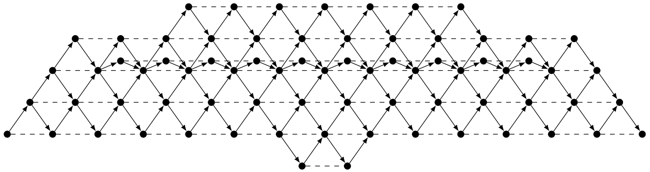
The AR-quiver of $2\text{-mod}\,\Lambda(6,4)$ is given by



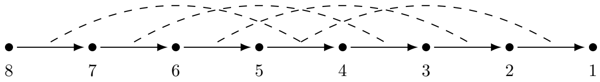
$2\text{-mod}\,\Lambda(7,4)$ – Python-assisted exploration



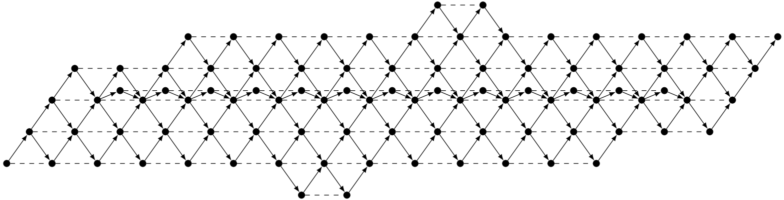
The AR-quiver of $2\text{-mod}\,\Lambda(7,4)$ is given by



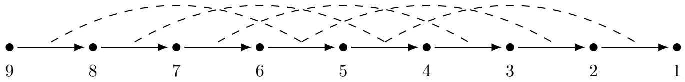
$2\text{-mod}\,\Lambda(8,4)$ – Python-assisted exploration



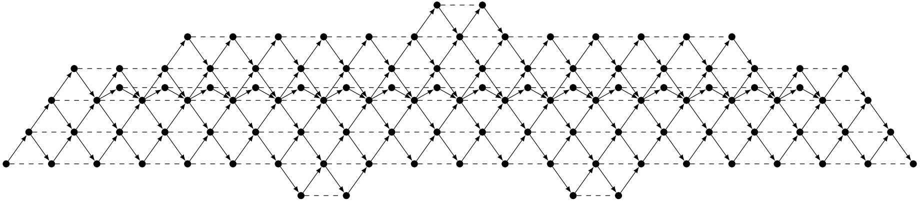
The AR-quiver of $2\text{-mod}\,\Lambda(8,4)$ is given by



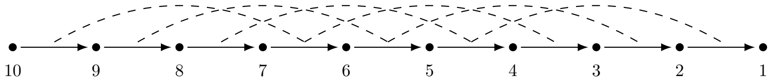
$2\text{-mod}\,\Lambda(9,4)$ – Python-assisted exploration



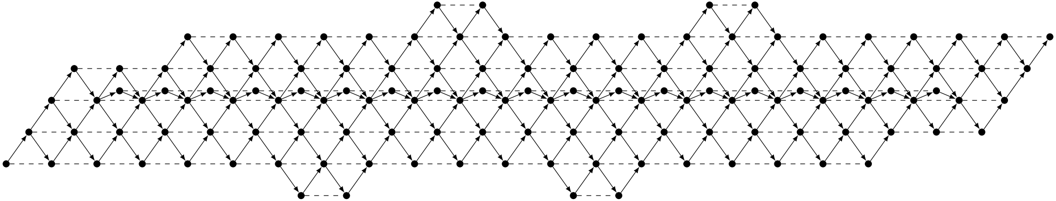
The AR-quiver of $2\text{-mod}\,\Lambda(9,4)$ is given by



$2\text{-mod}\,\Lambda(10,4)$ – Python-assisted exploration



The AR-quiver of $2\text{-mod}\,\Lambda(10,4)$ is given by



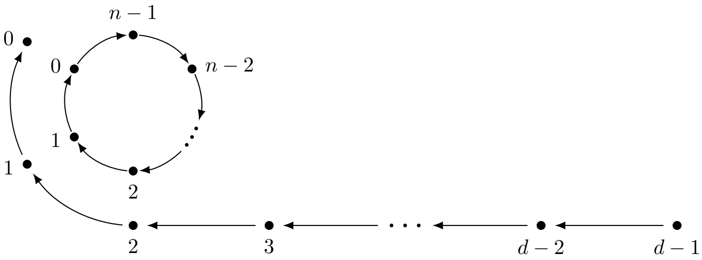
$m\text{-mod}\,\Lambda(n,l)$ – Finiteness

Teorem [R.'26]

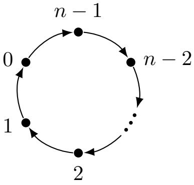
The extended module category $m\text{-mod}\,\Lambda(n,l)$ is of finite type exactly when it is contained in the table below.

$l \backslash n$	3	4	5	6	7	8	9	10	11	$\cdots$
2	All	All	All	All	All	All	All	All	All	$\cdots$
3		All	All	All	All	All	≤ 4	≤ 4	≤ 4	$\cdots$
4			All	All	All	≤ 2	≤ 2	≤ 2	≤ 2	$\cdots$
5				All	All	All	≤ 2	≤ 2	≤ 2	$\cdots$
6					All	All	$= 1$	$= 1$	$= 1$	$\cdots$
7						All	$= 1$	$= 1$	$= 1$	$\cdots$
$\vdots$							$\ddots$	$\ddots$	$\ddots$	$\ddots$

m-mod Λ – Cyclic cases



m-mod Λ – Cyclic cases



Teorem [R.'26]

Let Λ be a cyclic Nakayama algebra with n vertices, and relations given by all paths of length l. Then m-mod Λ is of finite type if and only if

1. $l = 2$,
2. $m \leq 4$ and $l = 3$,
3. $m \leq 2$ and either $l = 4$ or $l = 5$, or
4. $m = 1$.