NTNU | Department of Mathematical Sciences

THE CLASSIFICATION OF REPRESENTATION-FINITE SELFINJECTIVE ALGEBRAS

Endre Sørmo Rundsveen

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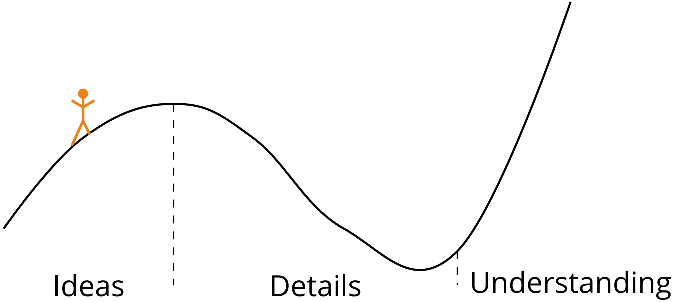
(My) Goal for this lecture



Fundamental notions

In this talk we assume that $\mathbf{k}$ is an algebraically closed field.

Definition

Let Λ be a finite dimensional $\mathbf{k}$-algebra.

1. Λ is *representation-finite*, if there are finitely many indecomposable Λ-modules up to isomorphism.
2. Λ is *self-injective* if the projective modules are also injective.

The classification program of Christine Riedtmann:

1979 Gabriel, Riedtmann. *Group representations without groups*

1980 Riedtmann. *Algebren, Darstellungsköcher, Überlagerungen und zurück*

1980 Riedtmann. *Representation-finite selfinjective algebras of class* A_n

1981 Bretscher, Läser, Riedtmann. *Self-injective and simply connected algebras*

1983 Riedtmann. *Representation-finite algebras of class* D_n

1983 Riedtmann. *Configurations of* $\mathbb{Z}D_n$.

Historical setting:

Beginning of 1970´s:

- Auslander–Reiten theory
- Quivers and tilting

Auslander–Reiten quivers, 1975 $\rightarrow$

Auslander–Reiten quivers

For a f.d. $\mathbf{k}$-algebra Λ, the Auslander-Reiten quiver Γ_Λ is a combinatorial description of the module category $\operatorname{mod}\Lambda$.

- Vertices are given by isoclasses of indecomposable modules,
- Arrows come from irreducible morphisms.

For each *almost split sequence*

$$0 \longrightarrow \tau M \longrightarrow \bigoplus E_i \longrightarrow M \longrightarrow 0$$

we have a *mesh*

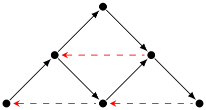
$$\begin{array}{ccccc} & \nearrow & [E_1] & \searrow & \\ & \nearrow & [E_2] & \searrow & \\ [\tau M] & & \vdots & & [M] \\ & \searrow & [E_r] & \nearrow & \end{array}$$

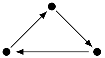
in Γ_Λ.

Examples

$\Lambda_1 = \mathbf{k}[\ \bullet \longrightarrow \bullet \longrightarrow \bullet\ ]$

$\Gamma_{\Lambda_1} =$



$\Lambda_2 = \mathbf{k}\Big[$  $\Big] \Big/ \langle \text{arrows} \rangle^4$

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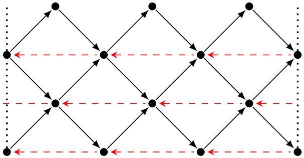


Translation quivers

Definition

Let Γ be a quiver without loops and double arrows. Assume that Γ is *locally finite*, i.e.

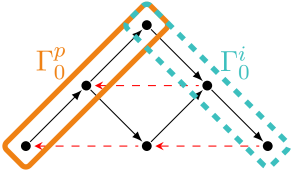
$$x^+ = \{\, y \,|\, \exists\, x \to y \,\} \quad \text{and} \quad x^- = \{\, z \,|\, \exists\, z \to x \,\}$$

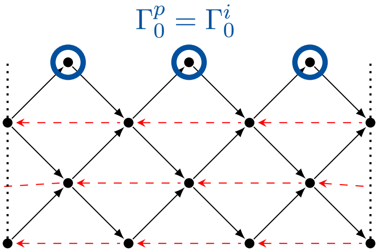
are finite for all $x \in \Gamma_0$. A *translation of* Γ is a bijection $\tau \colon \Gamma_0 \setminus \Gamma_0^p \to \Gamma_0 \setminus \Gamma_0^i$, for some subsets Γ_0^p and Γ_0^i of vertices, such that

$$(\tau x)^+ = x^- \forall x \in \Gamma_0.$$

The pair (Γ, τ) is a *translation quiver*.

Translation quivers





Definition

Let $\Gamma = (\Gamma, \tau)$ be a translation quiver. A vertex $x \in \Gamma_0$ is *stable* if $\tau^r x$ is defined for all $r \in \mathbb{Z}$. If all vertices are stable, we call Γ a stable translation quiver.

Every translation quiver Γ has a maximal stable subquiver ${}_s\Gamma$.

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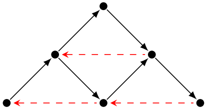

Γ_{Λ_1}

${}_s\Gamma_{\Lambda_1}$

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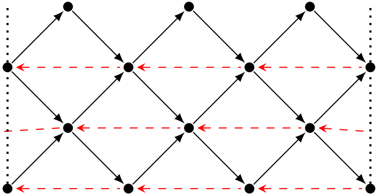
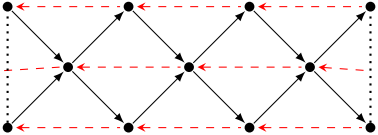
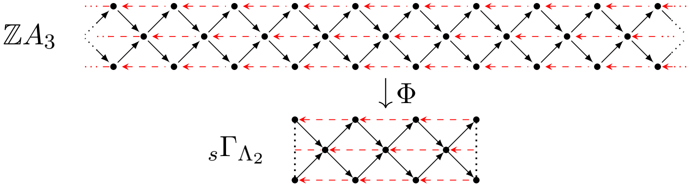


Γ_{Λ_2}



${}_s\Gamma_{\Lambda_2}$

Coverings



Definition

A morphism of translation quivers $\Phi\colon \tilde{\Gamma} \to \Gamma$ is a *covering* if

- $\Phi\colon \tilde{\Gamma}_0 \to \Gamma_0$ is surjective,
- $\forall x \in \tilde{\Gamma}$ the maps

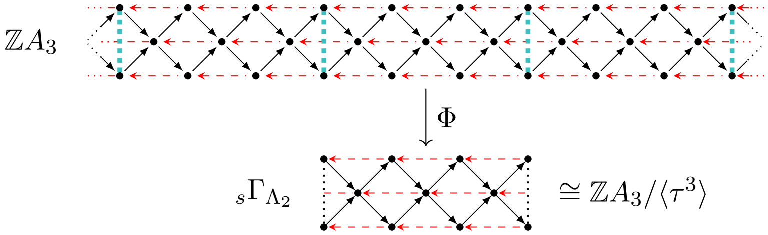
$$\Phi|_{x^+} : x^+ \to \Phi(x)^+ \quad \text{and} \quad \Phi|_{x^-} : x^- \to \Phi(x)^-$$

  are bijective.

For an *admissible* subgroup $\Pi \subseteq \mathrm{Aut}(\Gamma, \tau)$,

$$\pi : \Gamma \to \Gamma/\Pi$$

is a covering.



Riedtmann's Structure Theorem

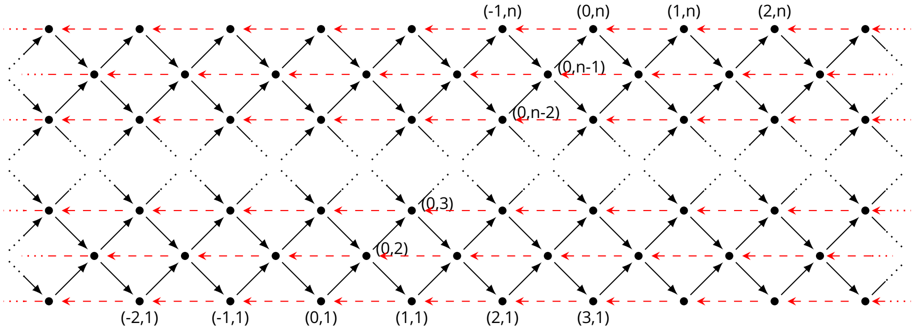
Teorem [Riedtmann]

Let Λ be a finite dimensional $\mathbf{k}$-algebra for finite representation type. Let Γ be a connected component of the stable AR-quiver ${}_s\Gamma_\Lambda$ of Λ. Then we have a universal covering

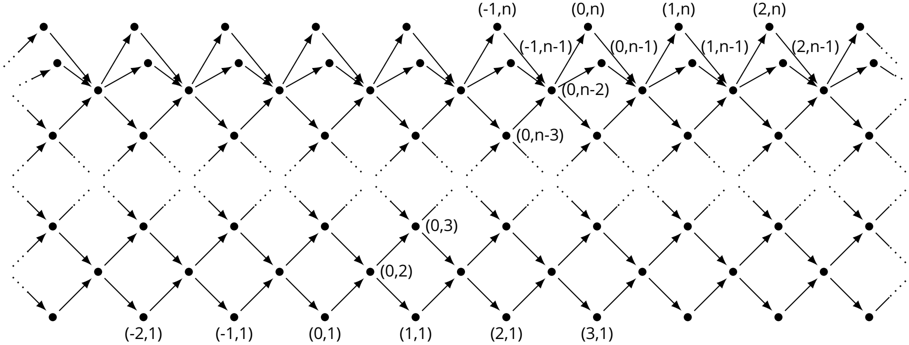
$$\mathbb{Z}\Delta/\Pi \to \Gamma$$

for $\Delta \in \{ A_n, D_n, E_6, E_7, E_8 \}$ and an admissible cyclic subgroup $\Pi \subseteq \mathrm{Aut}(\mathbb{Z}\Delta)$.

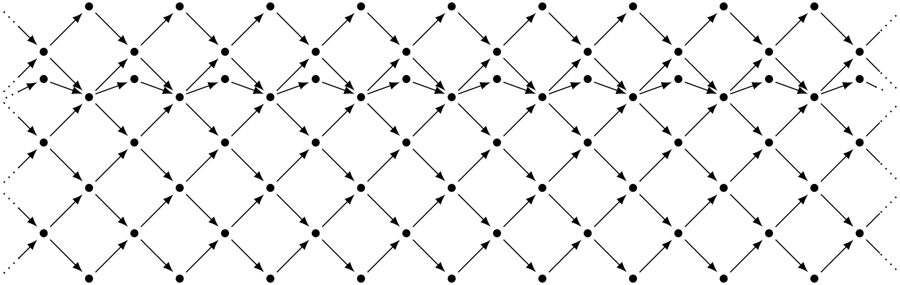
$\mathbb{Z}A_n$



$\mathbb{Z}D_n$



$\mathbb{Z}E_8$



Observation

- For Λ selfinjective, ${}_s\Gamma_\Lambda$ is obtained by deleting $[P]$ and the corresponding arrows for $P \in \operatorname{mod} \Lambda$ projective indecomposable.
- Moreover, for a projective and injective indecomposable module P, we have an almost split sequence

$$0 \longrightarrow \operatorname{rad} P \longrightarrow P \oplus (\operatorname{rad} P / \operatorname{Soc} P) \longrightarrow \operatorname{Soc} P \longrightarrow 0$$

Hence, for Λ selfinjective and rep-finite, ${}_s\Gamma_\Lambda$ is finite and connected.

Motivating conclusion

Teorem [Riedtmann]

Let Λ be a finite dimensional, selfinjective and representation-finite $\mathbf{k}$-algebra. Then we have a universal covering

$$\mathbb{Z}\Delta \to \mathbb{Z}\Delta/\Pi \cong {}_s\Gamma_\Lambda$$

for $\Delta \in \{ A_n, D_n, E_6, E_7, E_8 \}$ and an admissible cyclic subgroup $\Pi \subseteq \mathrm{Aut}(\mathbb{Z}\Delta)$.

Goal

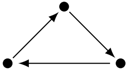
Determine the AR-quivers of rep-finite selfinjective algebras.

How

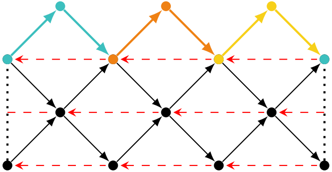
Knowing the stable part – How do we add projectives?

- Starting with an algebra, we simply add a vertex c^* and arrows $c \to c^*$ and $c^* \to \tau^{-1}c$ for each vertex $c = [\operatorname{rad} P]$ for P projective indecomposable.

Example

$\Lambda = \mathbf{k}\Big[$  $\Big] \Big/ \langle \text{arrows} \rangle^4$

$\Gamma = ({}_s\Gamma_\Lambda)_{\mathcal{C}}$



Let $\mathcal{C} = \{\, [\operatorname{rad} P] \mid P \in \operatorname{mod} \Lambda$ projective indecomposable $\}$.

Classification strategy

Find a set $\mathcal{C} \subseteq \mathbb{Z}\Delta$ and automorphisms Π such that $\mathbb{Z}\Delta_{\mathcal{C}}/\Pi$ is an AR-quiver for a self-injective rep-finite $\mathbf{k}$-algebra Λ.

$\mathbb{Z}\Delta_{\mathcal{C}}$ is obtained by adding

$$c \to c^* \quad \text{and} \quad c^* \to \tau^- c$$

to $\mathbb{Z}\Delta$ for each $c \in \mathcal{C}$.

Auslander Correspondence

Teorem **[Auslander]**

Up to Morita-equivalence, there is a bijection

$$\left\{ \begin{array}{c} \text{f.d. rep-finite} \\ \mathbf{k}\text{-algebras } \Lambda \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{f.d. } \mathbf{k}\text{-algebras } E \text{ s.t.} \\ \operatorname{gl.dim.} E \leq 2 \leq \operatorname{dom.dim} E \end{array} \right\}$$

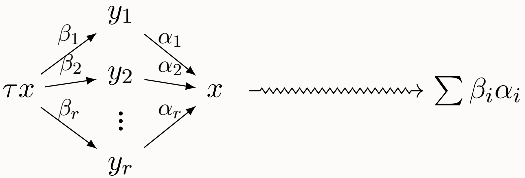
$$\Lambda \longmapsto E_\Lambda = \operatorname{End}_\Lambda\left(\bigoplus_{[M]\in \operatorname{ind} A} M\right)$$

Auslander Correspondence

Observe that the ordinary quiver of E_Λ, is simply the AR-quiver of Λ.

Definition

Let $\Gamma = (\Gamma, \tau)$ be a translation quiver. The *mesh algebra* $\mathbf{k}(\Gamma)$ is the bounded path algebra $\mathbf{k}\Gamma/I_\Gamma$, where I_Γ is the ideal generated by *mesh relations*, i.e. for each non-projective $x \in \Gamma$



Teorem [Riedtmann]

Let Λ be self-injective rep-finite such that

$$\Phi\colon \mathbb{Z}A_n \to {}_s\Gamma_\Lambda$$

is the universal covering with automorphism group Π. Let

$$\mathcal{C} = \Phi^{-1}(\{\,[\operatorname{rad} P] \mid P \in \operatorname{mod}\Lambda \text{ projective indecomposable}\,\}).$$

Then

$$\Gamma_\Lambda = (\mathbb{Z}A_n)_{\mathcal{C}}/\Pi,$$

and the Auslander algebra E_Λ is isomorphic to the mesh-algebra $\mathbf{k}(\Gamma_\Lambda)$.

Classification strategy – Modified

> Find $(\Delta, \Pi, \mathcal{C})$ such that $\mathbf{k}\,(\mathbb{Z}\Delta_{\mathcal{C}}/\Pi)$ is an Auslander algebra.

Such $\mathcal{C}$ are called *configurations* of $\mathbb{Z}\Delta$.

Combinatorial configurations

Definition

Let $\mathcal{C}$ be a set of vertices in $\mathbb{Z}\Delta$, $\Delta \in \{ A_n, D_n, E_6, E_7, E_8 \}$. $\mathcal{C}$ is a *combinatorial configuration* if

- for all vertices $x \in \mathbb{Z}\Delta$, there exists a vertex $c \in \mathcal{C}$ such that

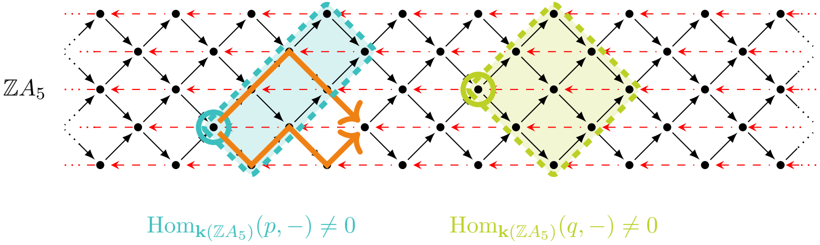
$$\mathrm{Hom}_{\mathbf{k}(\mathbb{Z}\Delta)}(x, c) \neq 0.$$

- for all pairs $c, d \in \mathcal{C}$ of distinct vertices,

$$\mathrm{Hom}_{\mathbf{k}(\mathbb{Z}\Delta)}(c, d) = 0.$$

Note, $\mathbf{k}(\mathbb{Z}\Delta)$ is here a category.

Combinatorial configurations – Type A_n



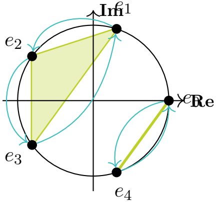
Combinatorial configurations – Type A_n

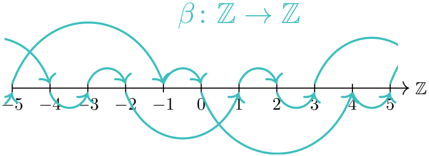
Proposition [Riedtmann]

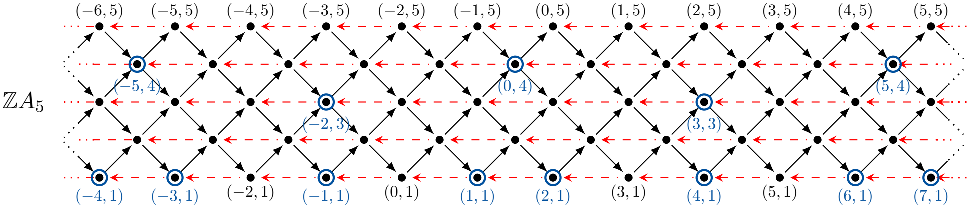
There is a bijection between the Brauer Relations of order n and the combinatorical configurations of $\mathbb{Z}A_n$.

A consequence of this is that configurations on $\mathbb{Z}A_n$ are periodic, $\tau^n \mathcal{C} = \mathcal{C}$.

Example – Combinatorial configuration







Admissible automorphisms

Remember the goal:

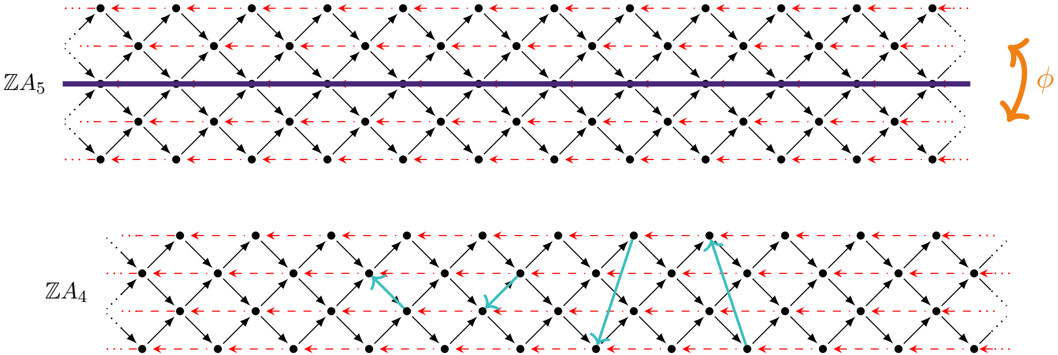
> Find $(\Delta, \Pi, \mathcal{C})$ such that $\mathbb{Z}\Delta_{\mathcal{C}}/\Pi$ is the AR-quiver for some selfinjective rep-finite $\mathbf{k}$-algebra Λ.

Any automorphism needs to stabilize $\mathcal{C}$.

Automorphisms of $\mathbb{Z}\Delta$ are of the form $(\tau^r\phi)^{\mathbb{Z}}$ [Riedtmann].

For A_n, ϕ is either

- $id_{\mathbb{Z}\Delta}$,
- a reflection, or
- a glide-reflection



$\mathbb{Z}A_5$
ϕ
$\mathbb{Z}A_4$

Admissible automorphisms

Proposition [Riedtmann]

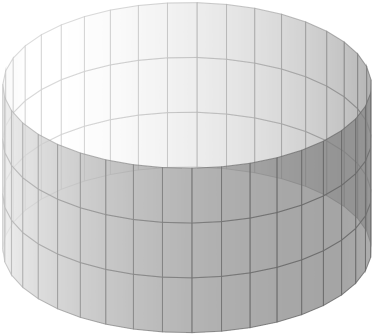
Assume there exists a reflection-translation ϕ of $\mathbb{Z}A_n$ which stabilizes the configuration $\mathcal{C}$. Then n is odd, $\mathcal{C}$ intersects the central line

$$\{(p, (n + 1/2)) \,|\, p \in \mathbb{Z}\},$$

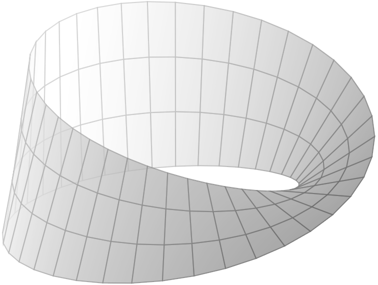
and the stabilizer of $\mathcal{C}$ is

$$\mathrm{Aut}(\mathbb{Z}A_n, \mathcal{C}) = \tau^{n\mathbb{Z}} \times \phi^{\mathbb{Z}/2\mathbb{Z}}.$$

Type A_n



$n \in \mathbb{Z}$, $\phi = \mathrm{id}_{\mathbb{Z}A_n}$



n odd, $\phi =$ reflection

Type A_n – Done

Teorem [Riedtmann]

Let $\mathcal{C}$ be a combinatorial configuration of $\mathbb{Z}A_n$ and Π an admissible group of automorphisms on $\mathbb{Z}A_n$ stabilizing $\mathcal{C}$. Then the mesh algebra $\mathbf{k}(\mathbb{Z}A_{n\mathcal{C}}/\Pi)$ is an Auslander algebra of a selfinjective algebra.

Remaining types

- Meant to follow a similar path for all Dynkin graphs.
- [Bretscher, Läser, Riedtmann] The configurations of $\mathbb{Z}\Delta$ correspond bijectively to isoclasses of $\vec{\Delta}$-section algebras.
- [Bretscher, Läser, Riedtmann] Description for type A_n can be lifted to remaining types.
- [Hughes, Waschbüsch] Each configuration of $\mathbb{Z}\Delta$ can be obtained from a tilting module over the path-algebra $\mathbf{k}\vec{\Delta}$.

Proposition [Bretscher, Läser, Riedtmann]

The **standard** rep-finite self-injective $\mathbf{k}$-algebras $\Lambda \neq \mathbf{k}$ are classified by the isomorphism classes

$$(\Delta, \Pi, \mathcal{C})$$

Troubles

- Underlying assumption was that a rep-finite selfinjective algebra is uniquely determined by its AR-quiver.

Definition

A rep-finite $\mathbf{k}$-algebra is *standard*, if $\mathbf{k}(\Gamma_\Lambda) \cong E_\Lambda$.

- Non-standard selfinjective exists for type $\Delta = D_{3m}$, $\mathrm{char}(\mathbf{k}) = 2$ and $\Pi = (\tau^{2m-1})^{\mathbb{Z}}$.